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THE FREE ALGEBRA IN THE SUBVARIETIES OF THE VARIETY OF REGULAR BANDS

Helmut LÄNGER
Institut für Algebra und Diskrete Mathematik, Technische Universität Wien, Austria

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A semigroup S is called a regular band if the laws $xx=x$ and $xyxzx=xyzx$ hold in S. For all subvarieties of the variety of regular bands a set-theoretic description of the free algebra is given. Moreover, it is proved that for fixed set of free generators these free algebras are subalgebras of the direct product of five elementary semigroups.

If one investigates free algebras it is sometimes useful to have a set-theoretic description of them avoiding terms, concatenation of terms and reduction of terms to 'normal form'. For some varieties of semigroups this was already done (cf. e.g. [1, 5, 6, 7, 8, 9]). The aim of this paper is to give such a description of the free algebras in the subvarieties of the variety of regular bands. Moreover, it is proved that for fixed set of free generators these free algebras are subalgebras of the direct product of five elementary semigroups.

Definition 1. A semigroup S is called a *band* if $xx=x$ for all $x\in S$. A band S is called *regular* if $xyxzx=xyzx$ for all $x,y,z\in S$.

Theorem 2 (cf. e.g. [2, 3, 4, 7]). *The lattice of all subvarieties of the variety of regular bands consists of the band varieties shown in Diagram* 1.

In the following let X be some fixed set. Since we will consider free groupoids on X and since in any groupoid variety the empty groupoid is free on $\emptyset$, we assume $X\neq\emptyset$. Moreover, in the following for any set M let 2^M denote the power set on M, let 2^X denote the semilattice $(2^X,\cup)$ and let 2^{2^X} denote the semigroup $(2^{2^X},\cdot)$ where

$$\mathfrak{M}\mathfrak{N}:=\mathfrak{M}\cup\{\bigcup\{M\mid M\in\mathfrak{M}\}\cup N\mid N\in\mathfrak{N}\}\quad(\mathfrak{M},\mathfrak{N}\in 2^{2^X}).$$

In order to see that the latter groupoid forms a semigroup, observe that

$$\bigcup\{P\mid P\in\mathfrak{M}\mathfrak{N}\}=\bigcup\{M\mid M\in\mathfrak{M}\}\cup\bigcup\{N\mid N\in\mathfrak{N}\}\quad\text{for all }\mathfrak{M},\mathfrak{N}\in 2^{2^X}$$

We are going to show that the free algebras on X in the subvarieties of the variety of

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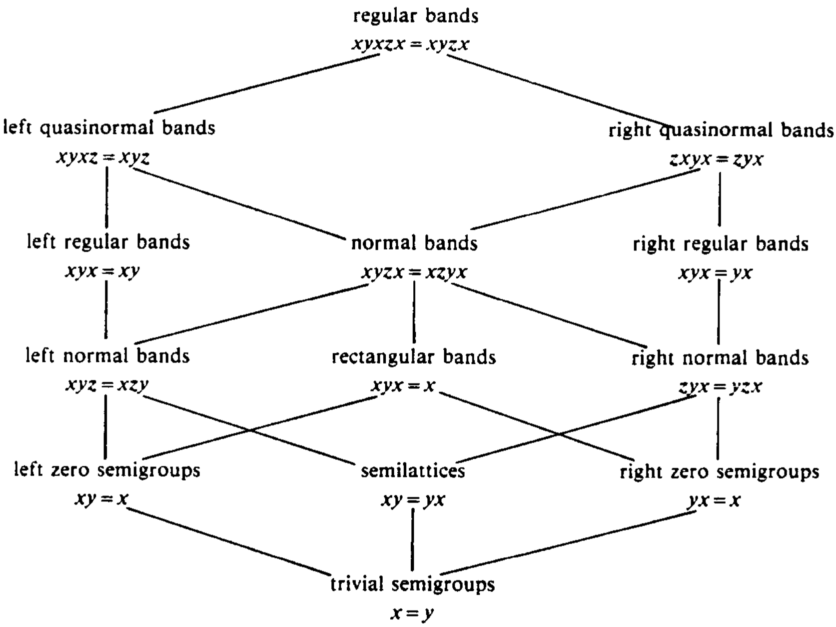

Diagram 1.

regular bands are subalgebras of the direct product of the following five semigroups: the left zero semigroup on X, the right zero semigroup on X, the semilattice 2^X, the semigroup 2^{2^X} and the dual of 2^{2^X}. In some of the considered varieties the following construction of the free algebra is well known (cf. e.g. [7]). For the sake of completeness we provide a complete list of results:

Theorem 3. *The one-element semigroup is the free trivial semigroup on X.*

Theorem 4. *Let F_{SL} denote the subalgebra $\{M \in 2^X \mid M \neq \emptyset,\ M \text{ finite}\}$ of 2^X and define $\iota : X \to F_{\mathrm{SL}}$ by $\iota(x) := \{x\}$ $(x \in X)$. Then (F_{SL}, ι) is the free semilattice on X.*

Theorem 5. *Let F_{LZ} denote the left zero semigroup on X and define $\iota : X \to F_{\mathrm{LZ}}$ by $\iota(x) := x$ $(x \in X)$. Then (F_{LZ}, ι) is the free left zero semigroup on X. The free right zero semigroup F_{RZ} on X is constructed dually.*

Theorem 6. *Let F_{RB} denote the direct product of F_{LZ} and F_{RZ} and define $\iota : X \to F_{\mathrm{RB}}$ by $\iota(x) := (x, x)$ $(x \in X)$. Then (F_{RB}, ι) is the free rectangular band on X.*

Theorem 7. *Let F_{LN} denote the subdirect product $\{(x, Y) \in F_{\mathrm{LZ}} \times F_{\mathrm{SL}} \mid x \in Y\}$ of F_{LZ} and F_{SL} and define $\iota : X \to F_{\mathrm{LN}}$ by $\iota(x) := (x, \{x\})$ $(x \in X)$. Then (F_{LN}, ι) is the free left normal band on X. The free right normal band F_{RN} on X is constructed dually.*

Theorem 8. *Let F_N denote the subdirect product $\{(x, Y, y) \in F_{LZ} \times F_{SL} \times F_{RZ} \mid x \in Y \ni y\}$ of F_{LZ}, F_{SL} and F_{RZ} and define $\iota: X \to F_N$ by $\iota(x) := (x, \{x\}, x)$ $(x \in X)$. Then (F_N, ι) is the free normal band on X.*

Theorem 9. *Let F_{LR} denote the subalgebra $\{\mathfrak{M} \in 2^{2^X} \mid$ there exists some finite non-empty subset Y of X such that $\mathfrak{M}$ is a maximal chain in $(2^Y, \subseteq)\}$ of 2^{2^X} and define $\iota: X \to F_{LR}$ by $\iota(x) := \{\emptyset, \{x\}\}$ $(x \in X)$. Then (F_{LR}, ι) is the free left regular band on X. The free right regular band F_{RR} on X is constructed dually.*

Proof. If Y_1, Y_2 are finite non-empty subsets of X, if $\mathfrak{M}_1$ is a maximal chain in $(2^{Y_1}, \subseteq)$ and if $\mathfrak{M}_2$ is a maximal chain in $(2^{Y_2}, \subseteq)$ then $\mathfrak{M}_1\mathfrak{M}_2$ is a maximal chain in $(2^{Y_1 \cup Y_2}, \subseteq)$. Hence, F_{LR} is a subalgebra of 2^{2^X}. Since

$$\mathfrak{M}\mathfrak{N}\mathfrak{P} = \mathfrak{M} \cup \{\bigcup\{M \mid M \in \mathfrak{M}\} \cup N \mid N \in \mathfrak{N}\}$$
$$\cup \{\bigcup\{M \mid M \in \mathfrak{M}\} \cup \bigcup\{N \mid N \in \mathfrak{N}\} \cup P \mid P \in \mathfrak{P}\}$$

for all $\mathfrak{M}, \mathfrak{N}, \mathfrak{P} \in F_{LR}$, F_{LR} is a left regular band. Moreover, $F_{LR} = \langle \iota X \rangle$. In order to prove freeness of (F_{LR}, ι) let f be some mapping from X to some left regular band S. Then the mapping $g: F_{LR} \to S$ defined by

$$g(\{\emptyset, \{x_1\}, \{x_1, x_2\}, \ldots, \{x_1, \ldots, x_n\}\}) := f(x_1) \cdots f(x_n)$$
$$(n \geq 1, x_1, \ldots, x_n \in X,\ x_1, \ldots, x_n \text{ mutually distinct})$$

is a homomorphism from F_{LR} to S satisfying $g\iota = f$. Applying a duality argument completes the proof of the theorem.

Theorem 10. *Let F_{LQN} denote the subdirect product*

$$\{(\mathfrak{M}, x) \in F_{LR} \times F_{RZ} \mid \bigcup\{M \mid M \in \mathfrak{M}\} \ni x\}$$

of F_{LR} and F_{RZ} and define $\iota: X \to F_{LQN}$ by $\iota(x) := (\{\emptyset, \{x\}\}, x)$ $(x \in X)$. Then (F_{LQN}, ι) is the free left quasinormal band on X. The free right quasinormal band F_{RQN} on X is constructed dually.

Proof. Since – by Theorem 9, Theorem 5 and Theorem 2 – F_{LR} and F_{RZ} are quasinormal bands, this is also true for F_{LQN}. Obviously, $F_{LQN} = \langle \iota X \rangle$. In order to prove freeness of (F_{LQN}, ι) let f be some mapping from X to some quasinormal band S. Then the mapping $g: F_{LQN} \to S$ defined by

$$g(\{\emptyset, \{x_1\}, \{x_1, x_2\}, \ldots, \{x_1, \ldots x_n\}\}, x) := f(x_1) \ldots f(x_n) f(x)$$
$$(n \geq 1, x_1, \ldots x_n, x \in X, x_1, \ldots x_n \text{ mutually distinct})$$

is a homomorphism from F_{LQN} to S satisfying $g\iota = f$. Applying a duality argument completes the proof of the theorem.

Theorem 11. *Let F_R denote the subdirect product*

$$\{(\mathfrak{M},\mathfrak{N})\in F_{LR}\times F_{RR} \mid \bigcup\{M \mid M\in\mathfrak{M}\}=\bigcup\{N \mid N\in\mathfrak{N}\}\}$$

of F_{LR} and F_{RR} and define $\iota: X\to F_R$ by $\iota(x):=(\{\emptyset,\{x\}\},\{\emptyset,\{x\}\})$ $(x\in X)$. Then (F_R,ι) is the free regular band on X.

Proof. From [5] it follows that (F_R,ι) is the free algebra on X in the semigroup variety V the corresponding set of equations of which is

$$\{x_1\ldots x_n=y_1\ldots y_m \mid n,m\geq 1,\ x_1,\ldots,x_n,y_1,\ldots,y_m\in X,$$
$$\{\emptyset,\{x_1\},\{x_1,x_2\},\ldots,\{x_1,\ldots,x_n\}\}=\{\emptyset,\{y_1\},\{y_1,y_2\},\ldots,\{y_1,\ldots,y_m\}\}$$
$$\{\emptyset,\{x_n\},\{x_n,x_{n-1}\},\ldots,\{x_n,\ldots,x_1\}\}=\{\emptyset,\{y_m\},\{y_m,y_{m-1}\},\ldots,\{y_m,\ldots,y_1\}\}\}.$$

Hence V is contained in the variety of regular bands. Because of Theorem 2 this inclusion is not proper.

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